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Stoke's theorem: Stoke's theorem provides a relation between line integral and surface integral.

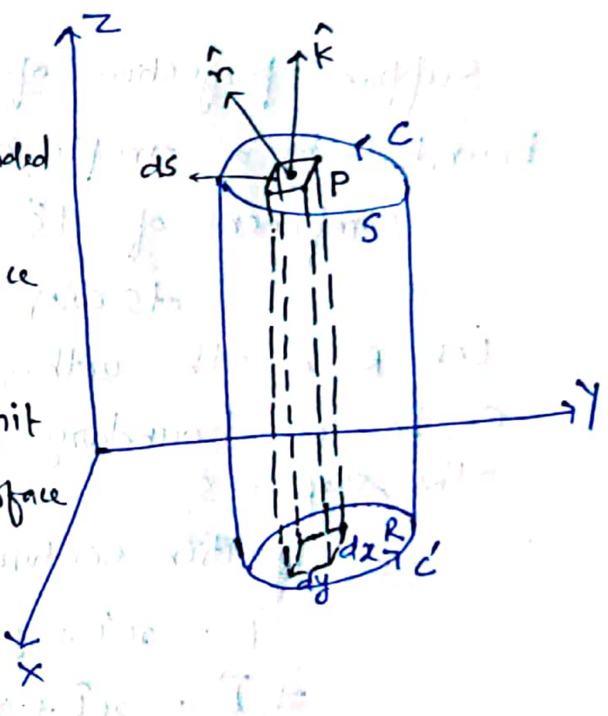
Statement of Stoke's theorem:– "Line integral of a vector point function $\vec{F}$ along any closed curve C is equal to surface integral of curl of the vector point function $\vec{F}$ over any surface S bounded by the closed curve C. When the function $\vec{F}$ should be continuous and differentiable over the surface S."

i.e, $$\oint_C \vec{F}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$

Where $\hat{n}$ is unit vector normal to the surface S.

proof :– Suppose $\vec{F}$ is a vector point function which is continuous and differentiable over a surface S bounded by a closed curve C.

P is a point on the surface S and ds be elementary area enclosing the point P. Let $\hat{n}$ be unit vector at point P normal to the surface S.



$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$d\vec{l} = dx\hat{i} + dy\hat{j} + dz\cdot\hat{k}$$

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Now $$\oint_C \vec{F}\cdot d\vec{l} = \oint_C (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$\oint_C \vec{F}\cdot d\vec{l} = \oint_C (F_1 dx + F_2 dy + F_3 dz) \quad \text{——} \quad (1)$$

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n} = \left\{\left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\cdot\frac{\delta}{\delta y} + \hat{k}\cdot\frac{\delta}{\delta z}\right)\times(F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\right\}\cdot(\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix} \cdot (\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$$

$$\Rightarrow (\vec{\nabla}\times\vec{F})\cdot\hat{n} = \left[\hat{i}\left(\frac{\delta F_3}{\delta y}-\frac{\delta F_2}{\delta z}\right)-\hat{j}\left(\frac{\delta F_3}{\delta x}-\frac{\delta F_1}{\delta z}\right)+\hat{k}\left(\frac{\delta F_2}{\delta x}-\frac{\delta F_1}{\delta y}\right)\right]\cdot(\cos\alpha\,\hat{i}+\cos\beta\,\hat{j}+\cos\gamma\,\hat{k})$$

$$=\left(\frac{\delta F_3}{\delta y}-\frac{\delta F_2}{\delta z}\right)\cos\alpha-\left(\frac{\delta F_3}{\delta x}-\frac{\delta F_1}{\delta z}\right)\cos\beta+\left(\frac{\delta F_2}{\delta x}-\frac{\delta F_1}{\delta y}\right)\cos\gamma$$

$$\Rightarrow(\vec{\nabla}\times\vec{F})\cdot\hat{n}=\left(\frac{\delta F_1}{\delta z}\cos\beta-\frac{\delta F_1}{\delta y}\cos\gamma\right)+\left(\frac{\delta F_2}{\delta x}\cos\gamma-\frac{\delta F_2}{\delta z}\cos\alpha\right)+\left(\frac{\delta F_3}{\delta y}\cos\alpha-\frac{\delta F_3}{\delta x}\cos\beta\right) \quad \text{——— (2)}$$

Using eqⁿ (1) and (2) in Stoke's theorem, we get

$$\oint_C F_1dx+F_2dy+F_3dz=\iint_S\left\{\left(\frac{\delta F_1}{\delta z}\cos\beta-\frac{\delta F_1}{\delta y}\cos\gamma\right)+\left(\frac{\delta F_2}{\delta x}\cos\gamma-\frac{\delta F_2}{\delta z}\cos\alpha\right)+\left(\frac{\delta F_3}{\delta y}\cos\alpha-\frac{\delta F_3}{\delta x}\cos\beta\right)\right\}ds. \quad \text{——— (A)}$$

Now we have to prove $\oint_C F_1dx=\iint_S\left(\frac{\delta F_1}{\delta z}\cos\beta-\frac{\delta F_1}{\delta y}\cos\gamma\right)ds$ ——— (3)

Suppose projection of surface S on xy plane is a region bounded by a simple closed curve.

Component of $d\vec{s}$ along z axis perpendicular to xy plane is

$$ds\cos\gamma=dx\cdot dy \Rightarrow ds=\frac{dx\,dy}{\cos\gamma} \quad \text{——— (4)}$$

Let R be the orthogonal projection of S on x-y plane and C_1 be its boundary and choosing $z=f(x,y)$ as the eqⁿ of the surface S.

position vector of the point P can be written as

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\Rightarrow\vec{r}=x\hat{i}+y\hat{j}+f(x,y)\hat{k} \quad \because z=f(x,y)$$

$$\Rightarrow\frac{d\vec{r}}{\delta y}=0+\frac{\delta y}{\delta y}\hat{j}+\frac{\delta f(x,y)}{\delta y}\cdot\hat{k}$$

$$\Rightarrow\frac{\delta\vec{r}}{\delta y}=\hat{j}+\frac{\delta f}{\delta y}\cdot\hat{k} \quad \because z=f(x,y)=f$$

where $\frac{d\vec{r}}{dy}$ = tangent vector at P

so $\frac{\delta\vec{r}}{\delta y}$ and $\hat{n}$ are perpendicular to each other

$$\Rightarrow\frac{\delta\vec{r}}{\delta y}\cdot\hat{n}=0\Rightarrow\left(\hat{j}+\frac{\delta f}{\delta y}\hat{k}\right)\cdot(\cos\alpha\,\hat{i}+\cos\beta\,\hat{j}+\cos\gamma\,\hat{k})=0$$

$$\Rightarrow\cos\beta+\cos\gamma\frac{\delta f}{\delta y}=0 \Rightarrow\cos\beta=-\cos\gamma\frac{\delta f}{\delta y} \text{ put in eq}^n\ (3)$$

$$\iint_S \left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) ds = \iint_S \left\{ \frac{\delta F_1}{\delta z}\left(-\cos\gamma \frac{\delta f}{\delta y}\right) - \frac{\delta F_1}{\delta y} \cos\gamma \right\} ds$$

$$= -\iint_S \left(\frac{\delta F_1}{\delta z}\cdot\frac{\delta f}{\delta y} + \frac{\delta F_1}{\delta y}\right)\cos\gamma . ds \quad \text{---} (5)$$

Now on surface S $F_1(x,y,z) = F_1\{x,y,f(x,y)\} = F(x,y)$ let

$\because z = f(x,y)$

$$\Rightarrow F(x,y) = F_1(x,y,z)$$

$$\frac{\delta F}{\delta y} = \frac{\delta F_1}{\delta x}\cdot\frac{\delta x}{\delta y} + \frac{\delta F_1}{\delta y}\cdot\frac{\delta y}{\delta y} + \frac{\delta F_1}{\delta z}\cdot\frac{\delta z}{\delta y}$$

$$= \frac{\delta F_1}{\delta x}\cdot 0 + \frac{\delta F_1}{\delta y}\cdot 1 + \frac{\delta F_1}{\delta z}\cdot\frac{\delta f}{\delta y} \quad \because z = f(x,y)$$

$$\Rightarrow \frac{\delta F}{\delta y} = \frac{\delta F_1}{\delta y} + \frac{\delta F_1}{\delta z}\cdot\frac{\delta f}{\delta y}$$

$$\Rightarrow \frac{\delta F_1}{\delta z}\cdot\frac{\delta f}{\delta y} + \frac{\delta F_1}{\delta y} = \frac{\delta F}{\delta y}$$ put in eqn (5), we get

$$\iint_S \left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) ds = -\iint_S \frac{\delta F}{\delta y}.\cos\gamma\, ds$$

$$\Rightarrow \iint_S \left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) ds = -\iint_R \frac{\delta F}{\delta y}.dx dy \quad \text{---} (6)$$ using eqn (4)

Now $$\oint_{C_1} F dx + 0.dy = \iint_R \left(\frac{\delta 0}{\delta x} - \frac{\delta F}{\delta y}\right) dx dy \Rightarrow \oint_{C_1} F dx = -\iint_R \frac{\delta F}{\delta y} dx dy$$ put in (6)

we get $$\iint_S \left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) ds = \oint_{C_1} F dx \quad \text{---} (7)$$

Since at each point (x,y) of C_1, the value of F will be same as the value of F_1 at each point (x,y,z) of C and dx is same for both curves, we must have

$$\oint_{C_1} F dx = \oint_C F_1 dx$$ put in eqn (7), we get

$$\iint_S \left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) ds = \oint_C F_1 dx \quad \text{---} (8)$$

Similarly we can prove ~~that~~

$$\iint_S \left(\frac{\delta F_2}{\delta x}\cos\gamma - \frac{\delta F_2}{\delta z}\cos\alpha\right) ds = \oint_C F_2\, dy \quad \text{——— (9)}$$

and $$\iint_S \left(\frac{\delta F_3}{\delta y}\cos\alpha - \frac{\delta F_3}{\delta x}\cos\beta\right) ds = \oint_C F_3\, dz \quad \text{——— (10)}$$

Adding eqn (8), (9) and (10), we get

$$\iint_S \left\{\left(\frac{\delta F_1}{\delta z}\cos\beta - \frac{\delta F_1}{\delta y}\cos\gamma\right) + \left(\frac{\delta F_2}{\delta x}\cos\gamma - \frac{\delta F_2}{\delta z}\cos\alpha\right) + \left(\frac{\delta F_3}{\delta y}\cos\alpha - \frac{\delta F_3}{\delta x}\cos\beta\right)\right\} ds$$

$$= \oint_C F_1\, dx + F_2\, dy + F_3\, dz$$

$$\Rightarrow \boxed{\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \oint_C \vec{F} \cdot d\vec{l}}$$ proved.

——— x ———